

# ON THE GEOMETRY OF THE $B$ -CONNECTION ON QUASI-KÄHLER MANIFOLDS WITH NORDEN METRIC

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## Abstract

The  $B$ -connection on almost complex manifolds with Norden metric is an analogue of the first canonical connection of Lihnerovich in Hermitian geometry. In the present paper it is considered a  $B$ -connection in the class of the quasi-Kähler manifold with Norden metric. Some necessary and sufficient conditions are derived for the corresponding curvature tensor to be Kählerian. Curvature properties for this connection are obtained. Conditions are given for the considered manifolds to be isotropic-Kähler.

**Key words:** almost complex manifold, Norden metric, nonintegrable structure,  $B$ -connection, quasi-Kähler manifolds

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## 1. PRELIMINARIES

Let  $(M, J, g)$  be a  $2n$ -dimensional *almost complex manifold with Norden metric*, i.e.  $M$  is a differentiable manifold with an almost complex structure  $J$  and a metric  $g$  such that

$$(1.1) \quad J^2x = -x, \quad g(Jx, Jy) = -g(x, y)$$

for arbitrary  $x, y$  of the algebra  $\mathfrak{X}(M)$  on the smooth vector fields on  $M$ .

The associated metric  $\tilde{g}$  of  $g$  on  $M$  is defined by  $\tilde{g}(x, y) = g(x, Jy)$ . Both metrics are necessarily of signature  $(n, n)$ . The manifold  $(M, J, \tilde{g})$  is an almost complex manifold with Norden metric, too.

Further,  $x, y, z, w$  will stand for arbitrary elements of  $\mathfrak{X}(M)$ .

A classification of the almost complex manifolds with Norden metric is given in [1]. This classification is made with respect to the tensor field  $F$  of type  $(0, 3)$  defined by

$$(1.2) \quad F(x, y, z) = g((\nabla_x J)y, z),$$

where  $\nabla$  is the Levi-Civita connection of  $g$ . The tensor  $F$  has the following properties

$$(1.3) \quad F(x, y, z) = F(x, z, y) = F(x, Jy, Jz).$$

Among the basic classes  $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$  of this classification, the almost complex structure is nonintegrable only in the class  $\mathcal{W}_3$ . This is the class of the

so-called *quasi-Kähler manifolds with Norden metric*, which we call briefly  *$\mathcal{W}_3$ -manifolds*. This class is characterized by the condition

$$(1.4) \quad \mathfrak{S}_{x,y,z} F(x, y, z) = 0,$$

where  $\mathfrak{S}$  is the cyclic sum over three arguments. The special class  $\mathcal{W}_0$  of the *Kähler manifolds with Norden metric* belonging to any other class is determined by the condition  $F(x, y, z) = 0$ .

Let  $R$  be the curvature tensor of  $\nabla$ , i.e.  $R(x, y)z = \nabla_x(\nabla_y z) - \nabla_y(\nabla_x z) - \nabla_{[x,y]}z$ . The corresponding tensor of type  $(0, 4)$  is determined by  $R(x, y, z, w) = g(R(x, y)z, w)$ .

The following Ricci identity for almost complex manifolds with Norden metric is known

$$(1.5) \quad (\nabla_x F)(y, z, w) - (\nabla_y F)(x, z, w) = R(x, y, Jz, w) - R(x, y, z, Jw).$$

The components of the inverse matrix of  $g$  are denoted by  $g^{ij}$  with respect to the basis  $\{e_i\}$  of the tangent space  $T_p M$  of  $M$  at a point  $p \in M$ .

The *square norm* of  $\nabla J$  is defined by

$$(1.6) \quad \|\nabla J\|^2 = g^{ij} g^{ks} g((\nabla_{e_i} J) e_k, (\nabla_{e_j} J) e_s).$$

In [2] the following equation is proved for a  $\mathcal{W}_3$ -manifold

$$(1.7) \quad \|\nabla J\|^2 = -2g^{ij} g^{ks} g((\nabla_{e_i} J) e_k, (\nabla_{e_s} J) e_j).$$

An almost complex manifold with Norden metric  $(M, J, g)$  is Kählerian iff  $\nabla J = 0$ . It is clear that we have  $\|\nabla J\|^2 = 0$  for such a manifold, but the inverse one is not always true. An almost complex manifold with Norden metric with  $\|\nabla J\|^2 = 0$  is called an *isotropic-Kählerian* in [2].

The Ricci tensor  $\rho$  for the curvature tensor  $R$  and the scalar curvature  $\tau$  for  $R$  are defined respectively by

$$(1.8) \quad \rho(x, y) = g^{ij} R(e_i, x, y, e_j), \quad \tau = g^{ij} \rho(e_i, e_j),$$

and their associated quantities  $\rho^*$  and  $\tau^*$  are determined respectively by

$$(1.9) \quad \rho^*(x, y) = g^{ij} R(e_i, x, y, J e_j), \quad \tau^* = g^{ij} \rho(e_i, J e_j).$$

Similarly, the Ricci tensor and the scalar curvature are determined for each *curvature-like tensor* (*curvature tensor*)  $L$ , i.e. for the tensor  $L$  with the following properties:

$$(1.10) \quad L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

$$(1.11) \quad \mathfrak{S}_{x,y,z} L(x, y, z, w) = 0 \quad (\text{first Bianchi identity}).$$

A curvature-like tensor is called a *Kähler tensor* if it has the property

$$(1.12) \quad L(x, y, Jz, Jw) = -L(x, y, z, w).$$

2. THE  $B$ -CONNECTION ON  $\mathcal{W}_3$ -MANIFOLDS

A linear connection  $D$  on an almost complex manifold with Norden metric  $(M, J, g)$  preserving  $J$  and  $g$ , i.e.  $DJ = Dg = 0$ , is called a *natural connection* [3]. In [4], on  $(M, J, g) \in \mathcal{W}_1$  a natural connection is considered defined by

$$(2.1) \quad D_x y = \nabla_x y + \frac{1}{2}(\nabla_x J)Jy,$$

which is called a  $B$ -connection. This connection is known in Hermitian geometry as the *first canonical connection of Lihnerovich*.

Let  $T$  be a torsion tensor of the  $B$ -connection  $D$  determined on  $(M, J, g)$  by (2.1). Because of the symmetry of  $\nabla$ , from (2.1) we have  $T(x, y) = \frac{1}{2}\{(\nabla_x J)Jy - (\nabla_y J)Jx\}$ . Then, having in mind (2.1), we obtain

$$T(x, y, z) = g(T(x, y), z) = \frac{1}{2}\{F(x, Jy, z) - F(y, Jx, z)\}.$$

Let us substitute  $z$  by  $Jz$ . Then, because of (1.3), the last equality implies

$$(2.2) \quad T(x, y, Jz) = \frac{1}{2}\{F(x, y, z) - F(y, x, z)\}.$$

We consider the  $B$ -connection  $D$  determined by (2.1) on a  $\mathcal{W}_3$ -manifold  $(M, J, g)$ . Then, because of (1.4), from (2.2) we have

$$(2.3) \quad \underset{x, y, z}{\mathfrak{S}} T(x, y, Jz) = 0.$$

Therefore, the proposition is valid.

**Proposition 2.1.** *The torsion tensor  $T$  of the  $B$ -connection  $D$  on a  $\mathcal{W}_3$ -manifold  $(M, J, g)$  satisfies the identity (2.3).*

Let  $Q$  be the tensor field determined by

$$(2.4) \quad Q(y, z) = \frac{1}{2}(\nabla_y J)Jz.$$

Having in mind (1.2), for the corresponding tensor field of type (0,3) we have

$$(2.5) \quad Q(y, z, w) = \frac{1}{2}F(y, Jz, w).$$

Because of the properties (1.3), (2.5) implies  $Q(y, z, w) = -Q(y, w, z)$ .

Let  $K$  be the curvature tensor of the  $B$ -connection  $D$ , i.e.  $K(x, y)z = D_x(D_y z) - D_y(D_x z) - D_{[x, y]}z$ . Then, according to (2.1) and (1.5), for the corresponding tensor of type (0, 4) we have

$$(2.6) \quad \begin{aligned} K(x, y, z, w) &= R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) \\ &\quad + Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \end{aligned}$$

After a covariant differentiation of (2.5), a substitution in (2.6), a use of (1.2), (1.3), (2.4) and some calculations, from (2.6) we obtain

$$K(x, y, z, w) = R(x, y, z, w) - \frac{1}{2} [(\nabla_x F)(y, z, Jw) - (\nabla_y F)(x, z, Jw)] \\ - \frac{1}{4} [g((\nabla_y J)z, (\nabla_x J)w) - g((\nabla_x J)z, (\nabla_y J)w)].$$

The last equality, having in mind (1.5), implies

$$(2.7) \quad K(x, y, z, w) = \frac{1}{4} \{2R(x, y, z, w) - 2R(x, y, Jz, Jw) + P(x, y, z, w)\},$$

where  $P$  is the tensor determined by

$$(2.8) \quad P(x, y, z, w) = g((\nabla_x J)z, (\nabla_y J)w) - g((\nabla_y J)z, (\nabla_x J)w).$$

In this way, the theorem is valid.

**Theorem 2.2.** *The curvature tensor  $K$  of the  $B$ -connection  $D$  on a  $\mathcal{W}_3$ -manifold  $(M, J, g)$  has the form (2.7).*

From (2.7) it follows immediately that the property (1.10) is valid for  $K$ . Because of  $DJ = Dg = 0$ , the property (1.12) for  $K$  is valid, too. Therefore, the property (1.11) for  $K$  is a necessary and sufficient condition  $K$  to be a Kähler tensor. Since  $R$  satisfies (1.11), then from (2.7) we obtain immediately the following

**Theorem 2.3.** *The curvature tensor  $K$  of the  $B$ -connection  $D$  on a  $\mathcal{W}_3$ -manifold  $(M, J, g)$  is Kählerian iff*

$$(2.9) \quad \underset{x, y, z}{\mathfrak{S}} R(x, y, Jz, Jw) = \underset{x, y, z}{\mathfrak{S}} P(x, y, z, w).$$

Let the following condition be valid for the  $\mathcal{W}_3$ -manifold  $(M, J, g)$

$$(2.10) \quad \underset{x, y, z}{\mathfrak{S}} R(x, y, Jz, Jw) = 0.$$

The condition (2.10) characterizes the class  $\mathcal{L}_2$  of the almost complex manifolds with Norden metric according to the classification in [5] with respect to the properties of  $R$ .

The equality (2.8) implies immediately the properties (1.10) and (1.12) for  $P$ . Then, according to (2.9) and (2.10), we obtain the following

**Theorem 2.4.** *Let  $(M, J, g)$  belongs to the class  $\mathcal{W}_3 \cap \mathcal{L}_2$ . Then the curvature tensor  $K$  of the  $B$ -connection  $D$  is Kählerian iff the tensor  $P$  determined by (2.8) is Kählerian, too.*

Having in mind (2.7), the last theorem implies the following

**Corollary 2.5.** *Let the curvature tensor  $K$  of the  $B$ -connection  $D$  be Kählerian on  $(M, J, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$ . Then the tensor  $H$ , determined by*

$$(2.11) \quad H(x, y, z, w) = R(x, y, z, w) - R(x, y, Jz, Jw)$$

*is a Kähler tensor.*

3. CURVATURE PROPERTIES OF THE CONNECTION  $D$  IN  $\mathcal{W}_3 \cap \mathcal{L}_2$ 

Let us consider the manifold  $(M, J, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$  with Kähler curvature tensor  $K$  of the  $B$ -connection  $D$ . Then, according to Theorem 2.4 and Corollary 2.5, the tensor  $P$  and  $H$ , determined by (2.8) and (2.11), respectively, are also Kählerian.

Let  $\rho(K)$  and  $\rho(P)$  be the Ricci tensors for  $K$  and  $P$ , respectively. Then we obtain immediately from (2.7)

$$(3.1) \quad \rho(y, z) - \rho^*(y, Jz) = 2\rho(K)(y, z) - \frac{1}{2}\rho(P)(y, z).$$

We denote  $\tau^{**} = g^{ij}g^{ks}R(e_i, e_k, J e_s, J e_j)$  and from (3.1) we have

$$(3.2) \quad \tau - \tau^{**} = 2\tau(K) - \frac{1}{2}\tau(P),$$

where  $\tau(K)$  and  $\tau(P)$  are the scalar curvatures for  $K$  and  $P$ , respectively.

It is known from [2], that  $\|\nabla J\|^2 = -2(\tau + \tau^{**})$ . Then (3.2) implies

$$(3.3) \quad \tau = \tau(K) - \frac{1}{4}(\tau(P) + \|\nabla J\|^2).$$

From (2.8) we obtain

$$\rho(P)(y, z) = g^{ij}g((\nabla_{e_i} J)z, (\nabla_{e_j} J)e_j),$$

from where

$$\tau(P) = g^{ij}g^{ks}g((\nabla_{e_i} J)e_s, (\nabla_{e_k} J)e_j).$$

Hence, applying (1.7), we get

$$(3.4) \quad \tau(P) = -\frac{1}{2}\|\nabla J\|^2.$$

From (3.3) and (3.4) it follows

$$(3.5) \quad \tau = \tau(K) - \frac{1}{8}\|\nabla J\|^2.$$

The equality (3.5) implies the following

**Proposition 3.1.** *Let the curvature tensor  $K$  of the  $B$ -connection  $D$  be Kählerian on  $(M, J, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$ . Then  $(M, J, g)$  is an isotropic-Kähler manifold iff  $\tau = \tau(K)$ .*

Let the considered manifold with Kähler curvature tensor  $K$  of the  $B$ -connection  $D$  in  $\mathcal{W}_3 \cap \mathcal{L}_2$  be 4-dimensional. Since  $H$  is a Kähler tensor, then according to [9], we have

$$(3.6) \quad H = \nu(H)(\pi_1 - \pi_2) + \nu^*(H)\pi_3,$$

where  $\nu(H) = \frac{\tau(H)}{8}$ ,  $\nu^*(H) = \frac{\tau^*(H)}{8}$  and

$$\begin{aligned}\pi_1(x, y, z, w) &= g(y, z)g(x, w) - g(x, z)g(y, w), \\ \pi_2(x, y, z, w) &= g(y, Jz)g(x, Jw) - g(x, Jz)g(y, Jw), \\ \pi_3(x, y, z, w) &= -g(y, z)g(x, Jw) + g(x, z)g(y, Jw), \\ &\quad - g(y, Jz)g(x, w) + g(x, Jz)g(y, w).\end{aligned}$$

From (3.6), (2.7) and (2.8) follows the next

**Proposition 3.2.** *Let the curvature tensor  $K$  of the  $B$ -connection  $D$  be Kählerian on a 4-dimensional  $(M, J, g) \in \mathcal{W}_3 \cap \mathcal{L}_2$  and  $P$  and  $H$  are determined by (2.8) and (2.11), respectively. Then  $H$  has the form*

$$H = \frac{4\tau(K) - \tau(P)}{16}(\pi_1 - \pi_2) + \frac{4\tau^*(K) - \tau^*(P)}{16}\pi_3.$$

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